

On the ambiguity of functions represented by divergent power series

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Abstract

Assuming the asymptotic character of divergent perturbation series, we address the problem of ambiguity of a function determined by an asymptotic power expansion. We consider functions represented by an integral of the Laplace-Borel type, with a curvilinear integration contour. This paper is a continuation of results recently obtained by us in a previous work. Our new result contained in Lemma 3 of the present paper represents a further extension of the class of contours of integration (and, by this, of the class of functions possessing a given asymptotic expansion), allowing the curves to intersect themselves or return back, closer to the origin. Estimates on the remainders are obtained for different types of contours. Methods of reducing the ambiguity by additional inputs are discussed using the particular case of the Adler function in QCD.

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1. Introduction

In 1952, Freeman Dyson obtained the famous result [16] that perturbation series in QED are divergent. During the subsequent decades, similar results have been obtained [23–25, 29, 35] for most of the physically interesting field theories and models in quantum physics (for a review, see [18, 19] and references therein). This result was a surprise and set a challenge for a radical reformulation of perturbation theory. Dyson's suggestion to regard a divergent perturbation series as asymptotic has been universally adopted. Now the problem is not whether a perturbation series is convergent or divergent, but rather whether or not, and under what conditions, it uniquely determines the expanded function. A crucial task is to find effective additional inputs that would be able to reduce or, if possible, remove the ambiguity. If all expansion coefficients are known, the series may determine the sought function even if it is not convergent, and may not do so even if it is convergent. This depends on additional conditions.

How to deal with divergent series and how to sum them, under what conditions a power series is able to determine uniquely the expanded function and how to give a series a precise meaning are problems of paramount importance in quantum theory. Power expansions are badly needed in physics but, to ensure that they have clear mathematical meaning, additional conditions are necessary, which are often difficult to fulfill.

We discuss here the ambiguities of perturbation theory stemming from the (assumed) asymptotic character of the series. We recall in section 2 the Lemma of Watson (calling it Lemma 1). Then, in section 3, we briefly recall our Lemma 2, which we obtained and proved in ref. [10] for curvilinear contours of integration. Section 4 is a new result: we obtain and prove Lemma 3, which deals with certain specific forms of curvilinear integration contours.

We shall use the following definition of asymptotic series. Given a point set \mathcal{S} having the origin as a point of accumulation, the power series $\sum_{n=0}^{\infty} F_n z^n$ is said to be asymptotic to the function $F(z)$ as $z \rightarrow 0$ on \mathcal{S} , if the set of functions $R_N(z)$,

$$R_N(z) = F(z) - \sum_{n=0}^N F_n z^n, \quad (1)$$

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satisfies the condition

$$R_N(z) = o(z^N) \quad (2)$$

for all $N = 0, 1, 2, \dots$, $z \rightarrow 0$ and $z \in \mathcal{S}$. The standard notation for an asymptotic series is:

$$F(z) \sim \sum_{n=0}^{\infty} F_n z^n, \quad z \in \mathcal{S}, \quad z \rightarrow 0. \quad (3)$$

The function $F(z)$ may be singular at $z = 0$. The coefficients F_n can be defined by

$$F_n = \lim_{z \rightarrow 0, z \in \mathcal{S}} \frac{1}{z^n} \left[F(z) - \sum_{k=0}^{n-1} F_k z^k \right], \quad n = 0, 1, 2, \dots \quad (4)$$

where $\sum_{k=0}^{n-1} F_k z^k = 0$ for $n = 0$ by definition. The prescription (4) makes sense whenever the asymptotic expansion exists: one can define F_n without using the n -th derivative of $F(z)$, $z \in \mathcal{S}$, which may not exist.

Let us recall that if the power series $\sum_{n=0}^{\infty} F_n z^n$ is convergent in a neighbourhood \mathcal{L} of the origin and if $F(z)$ is holomorphic and *equal* to $\sum_{n=0}^{\infty} F_n z^n$ in \mathcal{L} , then $F(z)$ is uniquely determined in \mathcal{L} . No additional input is needed, in contrast with the case that the series is asymptotic. Asymptoticity can be checked only if one knows both the expansion coefficients and the expanded function $F(z)$.

The ambiguity of a function given by an asymptotic series is illustrated by Watson lemma.

2. The lemma of Watson

Consider the following integral

$$\Phi_{0,c}^{(\alpha,\beta)}(\lambda) = \int_0^c e^{-\lambda x^\alpha} x^{\beta-1} f(x) dx, \quad (5)$$

where $0 < c < \infty$ and $\alpha > 0$, $\beta > 0$. Let $f(x) \in C^\infty[0, c]$ and $f^{(k)}(0)$ defined as $\lim_{x \rightarrow 0+} f^{(k)}(x)$. Let ε be any number from the interval $(0, \pi/2)$.

Lemma 1. (*G.N. Watson*) *If the above conditions are fulfilled, the asymptotic expansion*

$$\Phi_{0,c}^{(\alpha,\beta)}(\lambda) \sim \frac{1}{\alpha} \sum_{k=0}^{\infty} \lambda^{-\frac{k+\beta}{\alpha}} \Gamma\left(\frac{k+\beta}{\alpha}\right) \frac{f^{(k)}(0)}{k!} \quad (6)$$

holds for $\lambda \rightarrow \infty$, $\lambda \in S_\varepsilon$, where S_ε is the sectorial region

$$|\arg \lambda| \leq \frac{\pi}{2} - \varepsilon < \frac{\pi}{2}. \quad (7)$$

The expansion (6) can be differentiated with respect to λ any number of times.

For the proof see for instance [15, 17, 22].

It is worth mentioning that the region (7) is independent of α , β and c , and the expansion coefficients in (6) do not depend on c . Further, the factor $\Gamma\left(\frac{k+\beta}{\alpha}\right)$ in (6) makes the expansion coefficients grow faster with k than those of the Taylor series for $f(x)$. For $\alpha = \beta = 1$, $\Gamma\left(\frac{k+\beta}{\alpha}\right)$ in (6) cancels with the factorial $k!$ in the denominator.

The integral (5) reveals the large ambiguity of the resummation procedures having the same asymptotic expansion. No particular value c of the upper limit of integration can be a priori preferred.

Below we shall recall our Lemma 2 (stated and proved in ref. [10]) showing a set of plausible conditions under which the integration contour in the Laplace-Borel integral can be bent.

3. Bending the integration contour

Let $G(r)$ be a continuous complex function $G(r) = r \exp(ig(r))$, where $g(r)$ is a real-valued function given on $0 \leq r < c$, with $0 < c \leq \infty$. Assume that the derivative $G'(r)$ is continuous on the interval $0 \leq r < c$ and a constant $r_0 > 0$ exists such that

$$|G'(r)| \leq K_1 r^{\gamma_1}, \quad r_0 \leq r < c, \quad (8)$$

for a nonnegative K_1 and a real γ_1 .

Let the constants $\alpha > 0$ and $\beta > 0$ be given and assume that the quantities

$$A = \inf_{r_0 \leq r < c} \alpha g(r), \quad B = \sup_{r_0 \leq r < c} \alpha g(r) \quad (9)$$

satisfy the inequality

$$B - A < \pi - 2\varepsilon, \quad (10)$$

where $\varepsilon > 0$.

Let the function $f(u)$ be defined along the curve $u = G(r)$ and on the disc $|u| < \rho$, where $\rho > r_0$. Assume $f(u)$ to be holomorphic on the disc and measurable on the curve. Assume that

$$|f(G(r))| \leq K_2 r^{\gamma_2}, \quad r_0 \leq r < c, \quad (11)$$

hold for a nonnegative K_2 and a real γ_2 .

Define the function $\Phi_{b,c}^{(\alpha,\beta,G)}(\lambda)$ for $0 \leq b < c$ by

$$\Phi_{b,c}^{(\alpha,\beta,G)}(\lambda) = \int_{r=b}^c e^{-\lambda(G(r))^\alpha} (G(r))^{\beta-1} f(G(r)) dG(r). \quad (12)$$

This integral exists since we assume $f(u)$ measurable along the curve $u = G(r)$ and bounded by (11).

Lemma 2. *If the above assumptions are fulfilled, then the asymptotic expansion*

$$\Phi_{0,c}^{(\alpha,\beta,G)}(\lambda) \sim \frac{1}{\alpha} \sum_{k=0}^{\infty} \lambda^{-\frac{k+\beta}{\alpha}} \Gamma\left(\frac{k+\beta}{\alpha}\right) \frac{f^{(k)}(0)}{k!} \quad (13)$$

holds for $\lambda \rightarrow \infty$, $\lambda \in \mathcal{T}_\varepsilon$, where

$$\mathcal{T}_\varepsilon = \{\lambda : \lambda = |\lambda| \exp(i\varphi), \quad -\frac{\pi}{2} - A + \varepsilon < \varphi < \frac{\pi}{2} - B - \varepsilon\}. \quad (14)$$

We refer the reader to our recent paper [10], where Lemma 2 is proved. The aim of the present paper is to show that a further generalization is possible. We shall show in section 4 that Lemma 2 in Ref. [10] can be improved to apply to a wider class of curvilinear contours, including those that were mentioned in Remark 9 of ref. [10]. According to that Remark, the parametrization $G(r) = r \exp(ig(r))$ does not include contours that cross a circle centred at $r = 0$ either touching or doubly intersecting it, so that the derivative $G'(r)$ does not exist or is not bounded. In particular, this parametrization does not include the contours

- that, starting from the origin and reaching a value r_1 of r , return back to a certain value $r_2 < r_1$, which is closer to the origin, and
- whose one or several parts coincide with a part of a circle centred at the origin, and
- that have, at some point, their tangent perpendicular to the radius vector.

In the following section 4, we shall discuss a result that generalizes some features of Lemma 2 and, among others, cover also the two items mentioned above. For simplicity, we limit ourselves to $\alpha = \beta = 1$ and c finite. We shall call this new result Lemma 3.

4. Allowing the contour to circumscribe a circle or get closer towards origin

Let a complex function $G(s)$ be given. It is a function of a real parameter s on an interval $[0, c]$, $c < \infty$. Assume that $G(s)$ has continuous derivatives on $[0, c]$, $G(0) = 0$, $G(s) \neq 0$ for any $s > 0$. Let the function $f(u)$ be defined along the curve $u = G(s)$ and on the disc \mathcal{K} defined by $|u| < \rho$, where $\rho > 0$. Assume $f(u)$ to be holomorphic on \mathcal{K} and measurable and bounded on the curve. This implies

$$|f(G(s))| \leq K_2 \quad \text{for } s \in [0, c]. \quad (15)$$

We choose a number s_1 such that $0 < s_1 < c$ and $G(s)$ lies in \mathcal{K} for $s \in [0, s_1]$. Define

$$A = \inf_{s_1 \leq s \leq c} \arg(G(s)), \quad B = \sup_{s_1 \leq s \leq c} \arg(G(s)) \quad (16)$$

and assume that

$$B - A < \pi - 2\varepsilon \quad (17)$$

where $\varepsilon > 0$. Denote

$$\mathcal{U}_\varepsilon = \left\{ \lambda : -\frac{\pi}{2} - A + \varepsilon < \arg[\lambda] < \frac{\pi}{2} - B - \varepsilon \right\}. \quad (18)$$

Define the function

$$\Phi_{a,b}^{(G)}(\lambda) = \int_{s=a}^{s=b} f(G(s)) e^{-\lambda(G(s))} dG(s), \quad (19)$$

for $0 \leq a < b \leq c$, where the suppression of the labels α and β indicates that we have chosen $\alpha = \beta = 1$. Note that we introduce here s , a new real variable, which parametrizes the length of the integration contour and, unlike r , does not mean the distance from the origin.

Lemma 3. *If the above assumptions are fulfilled, then the asymptotic expansion*

$$\Phi_{0,c}^{(G)}(\lambda) \sim \sum_{k=0}^{\infty} \lambda^{-(k+1)} f^{(k)}(0) \quad (20)$$

holds for $\lambda \rightarrow \infty$, $\lambda \in \mathcal{U}_\varepsilon$.

Remarks:

1. The cone \mathcal{U}_ε (18) is maximal. It is proved in [10] that outside \mathcal{T}_ε Lemma 2 might not be fulfilled. The same argument can be applied in the case of Lemma 3.
2. If a curve is rectifiable and of finite length, then the value of s for a point of the curve can be defined as a function of the length of the curve from the origin to that point.

Proof: For a given N , $f(u)$ can be expressed inside the circle of radius $\rho' < \rho$ in the form

$$f(u) = \sum_{k=0}^N \frac{f^{(k)}(0)}{k!} u^k + r_N(u), \quad |r_N(u)| \leq C_N |u|^{N+1}. \quad (21)$$

Since the interval $[0, c]$ is compact and $G'(s)$ is continuous there exists a constant K_1 such that

$$|G'(s)| \leq K_1 \quad \text{for } s \in [0, c]. \quad (22)$$

Further, there exists a positive number η such that

$$|G(s)| > \eta \quad \text{for } s \in [s_1, c] \quad (23)$$

(note that $G(s) \neq 0$ in $[s_1, c]$ because $s_1 > 0$).

Let us define

$$\tilde{G}(s) = \frac{s}{s_1} G(s_1) \quad \text{for } 0 \leq s \leq s_1, \quad \tilde{G}(s) = G(s) \quad \text{for } s > s_1. \quad (24)$$

Since the curves $\tilde{G}(s), G(s)$ lie in \mathcal{K} for $0 \leq s \leq s_1$ (note that $f(u)$ is holomorphic in \mathcal{K}) and $\tilde{G}(0) = G(0), \tilde{G}(s_1) = G(s_1)$, the integrals of the function $f(u)e^{-\lambda u}$ along these curves on the interval $[0, s_1]$ equal each other. We have

$$\Phi_{0,c}^{(G)}(\lambda) = \Phi_{0,s_1}^{(\tilde{G})}(\lambda) + \Phi_{s_1,c}^{(G)}(\lambda). \quad (25)$$

Let us define

$$I_{a,b}^k(\lambda) = \int_a^b \left(\frac{s G(s_1)}{s_1} \right)^k \exp \{ -\lambda(s/s_1)G(s_1) \} \frac{1}{s_1} G(s_1) ds, \quad (26)$$

where the integrals run along the ray $(s/s_1)G(s_1)$. We obtain, using (24),

$$\Phi_{0,s_1}^{(\tilde{G})}(\lambda) = \sum_{k=0}^N I_{0,s_1}^k(\lambda) f^{(k)}(0)/k! + \int_{s=0}^{s=s_1} r_N(\tilde{G}(s)) \exp \{ -\lambda \tilde{G}(s) \} d\tilde{G}(s). \quad (27)$$

Let us first calculate the terms $I_{0,s_1}^k(\lambda)$. We have

$$I_{0,s_1}^k(\lambda) = I_{0,\infty}^k(\lambda) - I_{s_1,\infty}^k(\lambda). \quad (28)$$

To calculate the first term $I_{0,\infty}^k(\lambda)$, we shall use the well-known formula:

$$\int_0^\infty x^{\delta-1} \exp \{ -\mu x \} dx = \frac{1}{\mu^\delta} \Gamma(\delta) \quad (29)$$

for $\text{Re } \delta > 0, \text{Re } \mu > 0$. If we take $\delta = k+1, \mu = (\lambda/s_1)G(s_1)$, we obtain

$$I_{0,\infty}^k = \frac{1}{\lambda^{k+1}} \Gamma(k+1). \quad (30)$$

We shall show that, for the last term in (25), the following inequality holds:

$$|\Phi_{s_1,c}^{(G)}(\lambda)| \leq K_1 K_2 c \exp \{ -|\lambda| \eta \sin \varepsilon \}, \quad (31)$$

which is an exponential estimate.

Having chosen the cone \mathcal{U}_ε (18) and using (23), we obtain for $\lambda \in \mathcal{U}_\varepsilon$ the inequality

$$\text{Re}[\lambda G(s)] \geq |\lambda| |G(s)| \cos[\arg \lambda + \arg(G(s))] \geq |\lambda| \eta \sin \varepsilon. \quad (32)$$

Hence

$$|e^{-\lambda G(s)}| = e^{-\text{Re}[\lambda G(s)]} \leq e^{-|\lambda| \eta \sin \varepsilon}. \quad (33)$$

The inequalities hold for s from the interval $[s_1, c]$. Further,

$$|\Phi_{s_1,c}^{(G)}(\lambda)| \leq K_1 K_2 \int_{s_1}^c e^{-|\lambda| \eta \sin \varepsilon} ds. \quad (34)$$

This proves that the estimate (31) holds.

Now we shall deal with $I_{s_1,\infty}^k(\lambda)$ (see (28)). We have

$$|I_{s_1,\infty}^k(\lambda)| \leq \left(\frac{|G(s_1)|}{s_1} \right)^{k+1} \int_{s_1}^\infty s^k \exp \{ -|\lambda| s/s_1 |G(s_1)| \sin \varepsilon \} ds. \quad (35)$$

The right hand side can be rewritten

$$\frac{1}{|\lambda|^{k+1} (\sin \varepsilon)^{k+1}} \int_{|\lambda G(s_1)| \sin \varepsilon}^\infty y^k e^{-y} dy. \quad (36)$$

Certainly

$$|I_{s_1,\infty}^k(\lambda)| \leq \frac{K_{k,\delta}}{|\lambda|^{k+1} (\sin \varepsilon)^{k+1} (1-\delta)} \exp \{ -(1-\delta) |\lambda G(s_1)| \sin \varepsilon \} \quad (37)$$

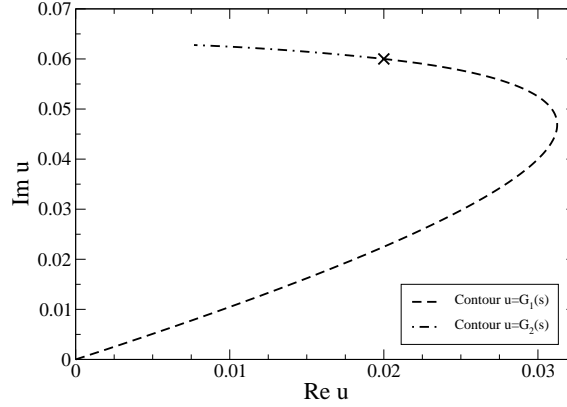


Figure 1: An integration contour allowed by Lemma 3, but which does not satisfy the conditions of Lemma 2. The cross marks the transition between the two curves discussed in the text.

for $|\lambda| > 1$, where $0 < \delta < 1$ and $y^k < K_{k,\delta} e^{\delta y}$ for $y > |G(s_1)| \sin \varepsilon$, which is again an exponential estimate.

The integral containing the remainder $r_N(z)$ (see (27)) can be estimated in a similar way using the inequality

$$\int_0^{s_1} (s |G(s_1)|/s_1)^{N+1} \exp \{-|\lambda|s/s_1 |G(s_1)| \sin \varepsilon\} 1/s_1 |G(s_1)| ds \leq \frac{\Gamma(N+2)}{|\lambda|^{N+2} (\sin \varepsilon)^{N+2}}, \quad (38)$$

which implies that the second term on the right hand side of (27) satisfies the inequality

$$\left| \int_0^{s_1} r_N(\tilde{G}(s)) \exp \{-\lambda \tilde{G}(s)\} d\tilde{G}(s) \right| \leq C_N \frac{\Gamma(N+2)}{|\lambda|^{N+2} (\sin \varepsilon)^{N+2}}. \quad (39)$$

This is a polynomial estimate of a lower degree than $I_{0,\infty}^k(\lambda)$.

5. Discussion

Lemma 3 and its proof cover up a set of integration contours that are not embraced in Lemma 2. In both cases, the contour of integration starts at the origin, $u = 0$. On the other hand, while the conditions of Lemma 2 admit only integration contours with increasing distance from the origin, the conditions of Lemma 3 permit a portion of the contour to get closer to the origin, or to have the form of an arc centred at the origin. Also, in Lemma 3, the integration contour may both perform spirals and intersect itself any number of times, with the reservation that the contour must not circle round the origin. It is a fundamental feature of both Lemma 2 and Lemma 3 that the integration contour of the Borel-Laplace integral must not leave the sectorial region \mathcal{T}_ε and \mathcal{U}_ε respectively.

To illustrate the above remarks we consider a simple example: let the curve $u = G_1(s)$ in the u -plane be defined parametrically by

$$\begin{aligned} G_1(s) &= t(s) + i v(s), & s &\in [0, 1], \\ t(s) &= a_1 s + a_2 s^2, & v(s) &= b_1 s + b_2 s^2, \end{aligned} \quad (40)$$

where a_1, a_2, b_1, b_2 are real parameters. It is easy to see that this curve satisfies the conditions of Lemma 3. On the other hand, it cannot be written always as $u = r \exp(ig(r))$, where $g(r)$ is a real function with a continuous first derivative, as requires Lemma 2. Indeed, let us make the change of variable

$$r \equiv r(s) = \sqrt{t(s)^2 + v(s)^2}. \quad (41)$$

Then

$$g(r) = \arctan[v(s(r))/t(s(r))], \quad (42)$$

where $s(r)$ is the inverse of (41). The derivative of (42) with respect to r can be written as

$$g'(r) = [v(s(r))t'(s(r)) - v'(s(r))t(s(r))] \frac{s'(r)}{r^2} \quad (43)$$

where $s'(r) = 1/r'(s)$. One can easily check that, for the choice $a_1 > 0$, $0 < b_1 < 2a_1$, and

$$a_2 = -\frac{3a_1 + b_1}{5}, \quad b_2 = \frac{a_1 - 3b_1}{5}. \quad (44)$$

one has $r'(s) > 0$ for $0 < s < 1$ and $r'(1) = 0$. Then, (42) is justified because $G_1(s)$ lies in the first quadrant.

It follows that $s'(r) \rightarrow \infty$ for $r \rightarrow r(1)$ and, since the first factor in (43) does not vanish at $r = r(1)$, $g'(1)$ is not bounded in the neighbourhood of $r = r(1)$, *i.e.* $g(r)$ does not fulfill the conditions of Lemma 2. (There are curves that possess infinitely many such points.)

The curve $G_1(s)$ can be further continued by $u = G_2(s)$ in such a way that the conditions of Lemma 3 are satisfied:

$$\begin{aligned} G_2(s) &= t(s) + i v(s), \quad s \in [1, 1.2], \\ t(s) &= (a_1 + a_2) \cos(s - 1) - (b_1 + b_2) \sin(s - 1), \quad v(s) = (a_1 + a_2) \sin(s - 1) + (b_1 + b_2) \cos(s - 1). \end{aligned} \quad (45)$$

For any values of a_i and b_i this curve is an arc of a circle centered at 0, therefore the derivative of $|G_2(s)|$ with respect to s is zero, while for the contours allowed in Lemma 2 the derivative should be equal to 1. In Fig. 1 we represent the union of the two curves discussed above, for the choice $a_1 = b_1 = 0.1$ and a_2, b_2 defined in (44).

6. Reducing the ambiguity by additional inputs

To discuss some physical applications we take the Adler function in massless QCD as an example. The Adler function $\mathcal{D}(s)$ (see [1]) is assumed to be real analytic in the complex s -plane cut along the timelike axis. The renormalization-group improved expansion,

$$\mathcal{D}(s) = D_1 \alpha_s(s)/\pi + D_2 (\alpha_s(s)/\pi)^2 + D_3 (\alpha_s(s)/\pi)^3 + \dots, \quad (46)$$

has an additional unphysical singularity due to the Landau pole in the running coupling $\alpha_s(s)$. According to present knowledge, (46) is divergent, the D_n growing as $n!$ at large n [4–6, 16, 26], see also [3, 20] and [19] and references therein.

To discuss the implications of Lemma 2 and Lemma 3, we define the Borel transform $B(u)$ by [27]

$$B(u) = \sum_{n \geq 0} b_n u^n, \quad b_n = \frac{D_{n+1}}{\beta_0^n n!}, \quad (47)$$

where β_0 is the first coefficient of the β function governing the renormalization group equation satisfied by the coupling. It is usually assumed that the series (47) is convergent on a disc of nonvanishing radius (this result was rigorously proved by David et al. [14] for the scalar φ^4 theory). This is exactly what is required in Lemmas 2 and 3 for the Borel transform.

If we adopt the assumption that the series (46) is asymptotic, both Lemma 2 and Lemma 3 imply a large freedom in recovering the true function from its perturbative coefficients. Indeed, taking for simplicity $\alpha = \beta = 1$ in (12), we infer that all the functions $\mathcal{D}_{0,c}^G(s)$ of the form

$$\mathcal{D}_{0,c}^G(s) = \frac{1}{\beta_0} \int_{r=0}^c e^{-\frac{G(r)}{\beta_0 a(s)}} B(G(r)) dG(r), \quad (48)$$

where $a(s) = \alpha_s(s)/\pi$, admit the asymptotic expansion of the type (46), provided that the assumptions of Lemma 3 are fulfilled. This reveals the large ambiguity of the resummation procedures having the same asymptotic expansion in perturbative QCD. No particular function of the form $\mathcal{D}_{0,c}^G(s)$ can be a priori preferred when looking for the true Adler function.

6.1. Mathematical conditions for uniqueness

For completeness, in this section we shall review several criteria for removing the ambiguity of a function represented by an asymptotic expansion. A powerful tool to reach uniqueness is provided by the Strong Asymptotic Conditions (SAC), which are conditions for Borel summability. The problem was investigated in many papers (see [34], [2], [31] and references therein).

The SAC are commonly used in two versions, one being due to G. Watson and another one due to F. Nevanlinna. Watson's version ([36], see also [34]) of the uniqueness criterion gives a sufficient condition for $F(z)$ to equal the Borel sum of its asymptotic Taylor series:

Watson's criterion: Assume $F(z)$ to be analytic in a sector $S_{\varepsilon,R}$, $|\arg z| < \pi/2 + \varepsilon$, $|z| < R$, for some positive ε , and let $F(z)$ have the asymptotic expansion

$$F(z) = \sum_{k=0}^{N-1} a_k z^k + R_N(z), \quad \text{where} \quad |R_N(z)| \leq A \sigma^N N! |z|^N \quad (49)$$

for $N = 0, 1, 2, \dots$ uniformly in N and in z in the sector. Then $F(z)$ is uniquely determined, being equal to

$$h(z) = \frac{1}{z} \int_0^\infty e^{-u/z} B(u) du, \quad \text{where} \quad B(u) = \sum_{n=0}^\infty \frac{a_n}{n!} u^n \quad (50)$$

inside the circle $\operatorname{Re} z^{-1} > 1/R$.

Note that ε is positive. This condition, sometimes difficult to satisfy, can be modified to a refined and improved version, which is due to Nevanlinna ([28], see also [34]). Nevanlinna's condition of Borel summability is:

Nevanlinna's criterion: Let $F(z)$ be analytic in the circle $C_R = \{z : \operatorname{Re} z^{-1} > 1/R\}$ and satisfy there the estimates (49) for $N = 0, 1, 2, \dots$ uniformly in N and in $z \in C_R$. Then $F(z)$ is uniquely determined and is equal to the function h defined in (50).

Nevanlinna's criterion gives both a sufficient and a necessary summability condition, see [28, 34]. Formally it is obtained from Watson's by replacing the sector $S_{\varepsilon,R}$ with the disc $C_{R'}$, where R and R' may be different.

In other words, among all the functions $F(z)$ analytic in C_R and possessing the asymptotic expansion (3) there is only one function, $h(z)$, which satisfies the inequalities (49) for all $N = 0, 1, 2, \dots$. Thus, among all functions $F(z)$ satisfying the expansion (3) there is one, $h(z)$, which is the best, in the sense that all the remainders $R_N(z)$, $N = 0, 1, 2, \dots$ are the smallest possible in C_R .

Further progress was achieved by T. Carleman [12]. Carleman's theorem can be used to show that two analytic functions with the same asymptotic expansion are identical. Some infinitely differentiable but non-analytic functions vanish identically in certain subsets of the complex plane. Carleman's theorem has the following form (see, e.g., [30]):

Carleman's theorem: Let g be a function analytic inside the sector $S_R = \{z, 0 \leq |z| \leq R, |\arg z| \leq \pi/2\}$ and continuous on S_R . Assume that

$$|g(z)| \leq b_N |z|^N \quad (51)$$

for every N and all $|z|$ inside the sector. If $\sum_{n=1}^\infty b_n^{-1/n} = \infty$, then g is identical zero.

The methods described above are effective ways to remove the infinite ambiguity by selecting the function $h(z)$, which is "the nearest" in the sense that the remainders $R_N(z)$ of all orders, $N = 0, 1, 2, \dots$, see (49) (or the function $g(z)$, see (51)), are the smallest possible in the respective region $S_{\varepsilon,R}$, C_R and S_R . Nearness is a natural criterion; on the other hand, it is not evident that nearness is always the best motivation from the physical point of view. It is worth discussing also other options.

6.2. Analyticity, its splendour and its dangerous points

In problems of divergence and ambiguity, the knowledge of the singularities of $\mathcal{D}(s)$ and of $B(u)$ is of importance. Some information about $B(u)$ follows from certain classes of Feynman diagrams, from renormalization theory and general nonperturbative arguments. Due to the singularities at u positive, the series (46) is not Borel summable. Except renormalons and instanton-antiinstanton pairs (i.e., $u \geq 2$ and $u \leq -1$), no other singularities of $B(u)$ are known. It is usually assumed that, with the exception of the above-mentioned singularities along the positive and the negative real semiaxes with a gap around the origin, $B(u)$ is holomorphic elsewhere.

To treat the analyticity properties of $B(u)$, the method of optimal conformal mapping [13] is very useful. Applications of this method to Lemma 2 and its merits are discussed in our previous paper [10]; the applications to Lemma 3 go along the same line. We refer the reader to [10] and references therein for details.

On the other hand, a careless manipulation with the integration contour may have a destructive effect on the analyticity properties. In [7, 21], two different contours are chosen for the summation of some class of diagrams: one contour, parallel and close to the positive semiaxis, is chosen for $a(s) > 0$, while another one, parallel and close to the negative semiaxis, is taken when $a(s) < 0$. As proved in [9], analyticity is lost with this choice, the summation being only piecewise analytic in s .

On the other hand, as shown in [8, 11], the Borel summation with the Principal Value (PV) prescription of the same class of diagrams admits an analytic continuation to the whole s -plane, being consistent with analyticity except for an unphysical cut along a segment of the space-like axis, related to the Landau pole. In this sense, PV is an appropriate prescription.

7. Concluding remarks

The main result of our work is Lemma 3 proved in section 4, which emphasizes the great ambiguity of functions represented by asymptotic power series. The result holds if the function $f(u)$ (which corresponds to the Borel transform) is analytic on a disc in the Borel plane and satisfies rather weak conditions outside the disc. Lemma 3 is an extension of Lemma 2 formulated and proved by us in ref. [10], and briefly mentioned in section 3 of the present paper. Lemma 2 and Lemma 3 are valid for two different classes of integration contours in the integral representations of the functions with a prescribed asymptotic expansion.

If applied to perturbation theory, Lemma 2 and Lemma 3 draw one's attention to the fact of a great ambiguity of the summation prescriptions that are allowed if the perturbation expansion is regarded as asymptotic. The contour of the integral representing the function of interest and the corresponding function $B(u)$ can be chosen very freely outside the convergence disc.

Lemma 2 and Lemma 3 proved in ref. [10] and, respectively, in this paper may also be useful in other branches of physics where the perturbation or other series are divergent.

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